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Vacuum Effects on Massive Spinor Fields: $S^1 \times R^3$ Topology

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The aim of this work was to investigate the role played by an external field on the Casimir energy density for massive fermions under $S^1 \times R^3$ topology. Both twisted- and untwisted-spin connections are considered and the calculation in a closed form is performed using an alternative approach based on the combination of the analytic regularization method and the Euler–Maclaurin summation formula. It is shown that no mass scale appears in the final result and, therefore, Casimir effect arises only from the boundary conditions and vacuum fluctuations induced by the coupling with the external field.

KEY WORDS: casimir effect; external field quantum electrodynamics.
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1. INTRODUCTION

The Casimir force between two parallel perfectly conducting plates is one of the most remarkable manifestations of quantum vacuum fluctuations. It was first predicted on theoretical grounds by Casimir (1948) and experimentally verified on a qualitative level by Sparnaay (1948) 10 years later. High accuracy experiments have been performed by Lamoreaux (1997) and by Mohideen and Roy (1998) and, more recently, by Bressi *et al*. (2002). For a more detailed account on the subject there are excellent reviews in the literature (Bordag *et al*., 2002; Mostepanenko and Trunov, 1997; Plunien *et al*., 1986).

Employing Casimir essential ideas, Johnson (1975) investigated the effects of boundaries on a massless Dirac field in the context of MIT-bag model (Chodos *et al*., 1974a,b) and found an energy density shift of the same order of magnitude as that obtained by Casimir for the electromagnetic field. Adopting Johnson's

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extended approach to the Casimir effect, in order to allow for other quantum fields, many authors have investigated the effects of different boundary conditions on the corresponding fields (DeWitt *et al.*, 1979). Hence, one can say that a modern view of the subject might take into account those effects due to nontrivial space topologies on the vacuum of relevant physical quantum fields.

It is worthwhile to note that, in this general context, some Casimir setups (field + boundary condition + external sources) present quite complicated final expressions for the Casimir energy density, which ultimately obscure any possible physical interpretation. In particular, the results obtained in Ford (1980) for the massive spinor field indicate a mass dependent energy density which calls for a deeper investigation. Similar difficulties also appear in the case considered in Cougo-Pinto *et al.* (2001) where the fermion field is also subjected to an external magnetic field.

In order to handle the above-mentioned shortcomings, we present here an alternative treatment that allows us to extract new information concerning the role of mass and that of an external field to the fermionic Casimir effect. This is achieved by a suitable combination of the method of analytic regularization using a suitable gamma function representation (also called α-representation (Bogoliubov and Shirkov, 1959)) and the well-known Euler–Mclaurin summation formula (Leavitt and Morrison, 1982).

The paper is organized as follows: In the next section the massive spinor field with both untwisted and twisted spin connections is considered. In Section 3, the effects of a weak external constant and homogeneous magnetic field is taken into account and the connection with the so-called Euler–Kockel–Heisenberg (E-K-H) Effective Lagrangian density (Euler, 1936; Euler and Kockel, 1935; Heisenberg and Euler, 1936) is addressed. Finally, in Section 4, we make some concluding remarks, pointing out directions of future investigations.

2. THE MASSIVE SPINOR FIELD: $S^1 \times R^3$

The case of noninteracting spinor fields subjected to $S^1 \times R^3$ space topology with twisted- and untwisted-spin connections was first considered by DeWitt *et al.* (1979) and, 1 year later, generalized to the massive case by Ford (1980), who obtained an intricate mass dependent expression for the vacuum energy density. In what follows we will re-derive these results using an alternative procedure.

2.1. Untwisted Case

Following Ford (1980), the vacuum energy density for the untwisted spinor field is given by

$$\varepsilon_0^{\text{unt}} = -\frac{2}{(2\pi)^2 L} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_x\, dp_y \left[m^2 + p_x^2 + p_y^2 + p_z^2\right]^{1/2}. \qquad (1)$$

where $p_z = a^2 n^2$, with $n = 0, \pm 1, \pm 2, \ldots$ and $a = 2\pi/L$.

In (1), the integrals are quadratically divergent quantities which claim for a consistent regularization prescription. Despite the familiar regularization methods found in the literature (Bogoliubov and Shirkov, 1959; Bonneau, 1990), we shall consider here yet another one, which proves to be useful. It consists in the combination of the analytic regularization scheme, using the gamma function integral representation, with the so-called Euler–Mclaurin summation formula (Leavitt and Morrison, 1982). To see how this works, we start by taking the analytic extension of the integrand in (1), which turns out to be a regular functional. This is achieved by means of the gamma function integral representation

$$\frac{1}{A^{1+\delta}} = \frac{1}{\Gamma(1+\delta)} \int_{0^+}^{\infty} d\eta\, \eta^{\delta}\, \mathrm{e}^{-A\eta}\ , \tag{2}$$

valid for $\delta > -1$, which allows us to rewrite (1) as

$$\begin{aligned}\left(\varepsilon_0^{\rm unt}\right)^R &= \frac{1}{2\pi L}\frac{1}{\Gamma(-1/2+\delta)} \sum_{n=-\infty}^{\infty} \int_{0^+}^{\infty} d\eta\, \eta^{-5/2+\delta} \mathrm{e}^{-(m^2+a^2n^2)\eta} \\ &= -\frac{1}{2\pi L}\frac{1}{\Gamma(-1/2+\delta)} \int_{0^+}^{\infty} d\eta\ \eta^{-5/2+\delta} \left[\mathrm{e}^{-m^2\eta} + 2\sum_{n=1}^{\infty} \mathrm{e}^{-(m^2+a^2n^2)\eta}\right],\end{aligned} \tag{3}$$

where we have already performed two gaussian integrals in p_x and p_y. Instead of using the Abel–Plana formula (Erdélyi *et al.*, 1953; Mostepanenko and Trunov, 1953), the divergent sum over n appearing in (3) will be performed by means of the Euler–Mclaurin summation formula (Leavitt and Morrison, 1982)

$$\begin{aligned}\sum_{n=M}^{N} f(n) &= \int_{M}^{N} F(x)dx + \frac{1}{2}[f(N)+f(M)] + \sum_{k=1}^{K} \frac{B_{2k}}{2k!}[F^{2k-1}(N) \\ &\quad - F^{2k-1}(M)] + \frac{1}{(2K+1)!}\int_{N}^{M} B_{2K+1}(x-[x])F^{2K+1}(x)\,dx\end{aligned} \tag{4}$$

where $B_m \equiv B_m(0)$, the $B_m(x)$ are the Bernoulli polynomials and F is an arbitrary function defined on the complex-plane such that, if n is integer and $M \leq n \leq N$, then $F(n) = f(n)$. We preserve here the same notation used in (Leavitt and Morrison, 1982). The last term in (4), also called the remainder term, vanishes if $F(z)$ is an entire function. In the present context, identifying the entire function $f(n)$ with

$$f(n) = \mathrm{e}^{-(m^2+a^2n^2)\eta}. \tag{5}$$

and, since $0 \leq n \leq \infty$, we are allowed to rewrite (3) as

$$(\varepsilon_0)^{\mathrm{R}} = \frac{1}{(2\pi)L} \frac{\pi}{\Gamma(-1/2+\delta)} \left\{ \int_{0^+}^{\infty} d\eta\, \eta^{-5/2+\delta} \left[\int_0^{\infty} dn\ f(n) - \frac{1}{2} f(n) \right.\right.$$
$$\times\ |_{n\to\infty} \frac{1}{12} f'(n)|_{n\to\infty} - \frac{1}{12} f'(n)|_{n\to 0} - \frac{1}{720} f'''(n)$$
$$\left.\left. \times\ |_{n\to\infty} + \frac{1}{720} f'''(n)|_{n\to 0} + \cdots \right]\right\} \tag{6}$$

where $f'(n)$ means the first derivative of (5) with respect to n and so on. Calculating the derivatives and taking the corresponding limits we see that the only nonvanishing terms give

$$\left(\varepsilon_0^{\mathrm{unt}}\right)^R = \frac{1}{(2\pi)L} \frac{\pi}{\Gamma(-1/2+\delta)} \left\{ \int_{0^+}^{\infty} d\eta\, \eta^{-3+\delta} \mathrm{e}^{-m^2\eta} \frac{\sqrt{\pi}}{2a} \right.$$
$$+ \frac{1}{12} \left(\frac{\Gamma(-1/2+\delta)}{(m^2+a^2n^2)^{-1/2+\delta}} (-2a^2 n) \right)_{n\to\infty}$$
$$\times\ \frac{1}{720} \left(\frac{\Gamma(-3/2+\delta)}{(m^2+a^2n^2)^{3/2+\delta}} (-2a^2 n)^3 \right)_{n\to\infty}$$
$$\left. -\ \frac{3}{720} \left(\frac{\Gamma(1/2+\delta)}{(m^2+a^2n^2)^{1/2+\delta}} (-2a^2)^2 n \right)_{n\to\infty} \right\}. \tag{7}$$

In going from expression (6) to (7) we notice that only the ($n \to \infty$)-terms contribute to the energy density. Usually, the methods found in the literature extract information arising from the ($n \to 0$)-terms and, as we will see below, this generates quite different final results. It must be stressed that, while the exponential function in (5) is an analytical function over the entire complex plane, the power function in the integrand of (1) is a multivalued function, which has a branch cut along the real axis (Courant and John, 1974). Further, analyzing the structure of (7), we see that the fermion mass appearing in the denominator of the last three terms may be neglected since in those terms $n \to \infty$. This point is crucial since the limit accounts for the partial elimination of the fermion mass. In fact, m remains in the kernel of the first term but, as will be shown later, this term will be cancelled against the Minkowiski vacuum energy.

In order to be consistent with the original theory we now take the limit $\delta \to 0$ in (7). However, we must first handle the divergent contribution arising from the second term in the curly brackets. This is done through the freedom in the choice of δ in (7). In fact, due (3), δ are constrained to be greater than 1/2. Thus, to obtain the correct final result, the considered region in the complex plane must be analytic continued to $\delta \geq 1$. As a result we have

$$\varepsilon_0^{\text{unt}} = \frac{1}{8\pi^2} \int_0^\infty d\eta\, \eta^{-3} e^{-m^2\eta} - \frac{\pi^2}{720L^4}. \tag{8}$$

Since we are dealing with negative energies (associated to the fermionic vacuum) we define the Casimir energy difference by subtracting (8) from the corresponding usual Minkowiski vacuum energy (which corresponds to taking $L \to \infty$ in (8)). In this way, we find the fermionic Casimir energy density

$$\Delta\varepsilon^{\text{unt}} = \frac{2\pi^2}{45L^4}, \tag{9}$$

which is in complete agreement with (De Witt *et al.*, 1979), the only difference being a factor 2, which reflects the four-component spinor field representation we are using. De Witt *et al.* (1979) considered a two-component spinor field, instead.

2.2. Twisted Case

Expression (14) clearly shows the independence of the Casimir energy density with respect to the fermion mass. This feature also occurs when twisted boundary condition is considered. In this case $p_z = (2n+1)a$, with $n = 0, \pm 1, \pm 2, \ldots$ and $a = \pi/L$. The regulated vacuum energy density is now given by

$$\left(\varepsilon_0^{\text{twi}}\right)^R = \frac{\sqrt{\pi}}{(2\pi)^2 L} \sum_{n=-\infty}^{\infty} \int_0^\infty d\eta\, \eta^{-5/2} \mathrm{e}^{-(m^2+(2n+1)^2a^2)\eta}. \tag{10}$$

Since the above integral is regular, we are allowed to interchange the sum and the integral, and then use the obvious mathematical trick

$$\sum_{n=-\infty}^{\infty} e^{-(m^2+(2n+1)^2a^2)\eta} = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-(m^2+n^2a^2)\eta} - \sum_{n=-\infty}^{\infty} e^{-(m^2+(2n)^2a^2)\eta}. \tag{11}$$

Thus, the problem of solving the twisted case reduces to that of computing two terms proportional to that in the untwisted case. In fact, we have

$$\begin{aligned}\varepsilon_0^{\text{twi}} &= \frac{1}{2^3}\varepsilon_0^{\text{unt}} - \varepsilon_0^{\text{unt}} \\ &= -\frac{7}{16(2\pi)^2} \int_{0^+}^\infty d\eta\, \eta^{-3} e^{-m^2\eta} + 2\frac{7\pi^2}{360L^4}.\end{aligned} \tag{12}$$

Again, subtracting this result from that where $L \to \infty$, we obtain

$$\Delta\varepsilon_0^{\text{twi}} = -2\frac{7\pi^2}{360L^4}, \tag{13}$$

which coincides with the familiar result found in the literature for the massless fermionic Casimir effect (DeWitt *et al.*, 1979).

3. THE INFLUENCE OF AN EXTERNAL FIELD

Another interesting problem to be analyzed using the present construct is related to the influence of an external magnetic field on the Casimir energy associated with the Dirac field. This problem has been recently proposed in the context of Effective Quantum Electrodynamics using the so-called Schwinger proper-time method (Cougo-Pinto *et al.*, 2001). However, a clear answer concerning the role of the external field on the fermionic Casimir energy density is yet an open problem which deserves further investigation. The purpose of this section is to implement, in the context of Weisskopf method (Weisskopf *et al.*, 1936; Tomazelli and Costa, 2003), the prescription presented in the previous sections in order to get a better understanding of the above mentioned problem.

We restrict our calculation to the case where an untwisted massless spinor field is subjected to an external weak and constant uniform magnetic field. As is well known (Berestetskii *et al.*, 1982), the negative energy levels for an electron of charge $e = -|e|$ in the presence of an uniform and constant magnetic field $H_z = -H$ is given by

$$-\epsilon^{(-)}_{\mathrm{p},\sigma} = -\sqrt{(2n+1-\sigma)|e|H + p_z^2}, \tag{14}$$

where $n = 0, 1, 2, 3\ldots$ and $\sigma = \pm 1$. Taking into account the density of states in the interval dp_z

$$\frac{|e|H}{2\pi}\frac{dp_z}{2\pi} \tag{15}$$

and the fact that all the levels except $n = 0, \sigma = 1$ are doubly degenerate (the levels $n, \sigma = -1$ and $n + 1, \sigma = 1$ coincide), we obtain the energy density of vacuum electrons,

$$\begin{aligned}\varepsilon'_0 &= -\sum_{\mathrm{p},\sigma} \epsilon^{(-)}_{\mathrm{p},\sigma} \\ &= -\frac{|e|H}{2\pi^2}\int_{-\infty}^{+\infty}\left\{\sqrt{p_z^2} + 2\sum_{n=1}^{\infty}\sqrt{2|e|Hn + p_z^2}\right\} dp_z. \end{aligned} \tag{16}$$

where p_z turned out to be a discrete quantity in virtue of the untwisted $S^1 \times R^3$ space topology we are assuming. Using (2), the energy density (16) may be rewritten in the more convenient form,

$$(\varepsilon'_0)^R = -\frac{|e|H}{2\pi L}\sum_{n=0}^{\infty}\frac{1}{\Gamma(-1/2+\delta)}\int_{0^+}^{\infty} d\eta\, \eta^{-3/2+\delta}\left(\sum_{n'=-\infty}^{\infty} \mathrm{e}^{-(2|e|Hn + a^2 n'^2)\eta}\right)$$

$$= -\frac{|e|H}{2\pi L}\sum_{n=0}^{\infty}\frac{1}{\Gamma(-1/2+\delta)}\int_{0^+}^{\infty} d\eta\ \eta^{-3/2+\delta}\mathrm{e}^{-\alpha\eta}\left(f(0)+2\sum_{n'=1}^{\infty}f(n')\right) \tag{17}$$

with $2|e|Hn = \alpha(n) \equiv \alpha$,

$$f(n') = \mathrm{e}^{-(an')^2\eta}, \tag{18}$$

and $n' = 0, \pm 1, \pm 2, \ldots, a = 2\pi/L$. Applying the Euler–Maclaurin formula (4) and performing the corresponding derivatives and limits, we arrive at

$$\begin{aligned}(\varepsilon_0')^R = &-\frac{\sqrt{\pi}}{(2\pi)^2}\frac{|e|H}{\Gamma(-1/2+\delta)}\int_{0^+}^{\infty} d\eta\ \eta^{-2+\delta}\sum_{n=0}^{\infty}\mathrm{e}^{-\alpha\eta}\\ &+\frac{2\pi}{3L^3}\frac{|e|H}{\Gamma(-1/2+\delta)}\left\{n'\int_{0^+}^{\infty} d\eta\ \eta^{-1/2+\delta}\mathrm{e}^{-(an')^2\eta}\sum_{n=0}^{\infty}\mathrm{e}^{-\alpha\eta}\right\}_{n'\to\infty}\\ &-\frac{2^4(2\pi)^5}{720L^7}\frac{|e|H}{\Gamma(-1/2+\delta)}\left\{n'^3\int_{0^+}^{\infty} d\eta\ \eta^{3/2+\delta}\mathrm{e}^{-(an')^2\eta}\sum_{n=0}^{\infty}\mathrm{e}^{-\alpha\eta}\right\}\\ &+\frac{12(2\pi)^3}{720L^5}\frac{|e|H}{\Gamma(-1/2+\delta)}\left\{n'\int_{0^+}^{\infty} d\eta\ \eta^{1/2+\delta}\mathrm{e}^{-(an')^2\eta}\sum_{n=0}^{\infty}\mathrm{e}^{-\alpha\eta}\right\}_{n'\to\infty}\end{aligned} \tag{19}$$

The sum in the integrands can be eliminated by noticing that

$$\sum_{n=2}^{\infty}\mathrm{e}^{-\alpha\eta} = \sum_{n=2}^{\infty}\mathrm{e}^{-2|e|H\eta} = \coth(|e|H\eta). \tag{20}$$

Furthermore, assuming the weak field regime, we are allowed to expand the kernel of the integrals in (19), namely,

$$\begin{aligned}(\varepsilon_0')^R = &-\frac{\sqrt{\pi}}{(2\pi)^2}\frac{|e|H}{\Gamma(-1/2+\delta)}\int_{0^+}^{\infty} d\eta\ \eta^{-2+\delta}\coth(|e|H\eta)\\ &+\frac{2\pi}{3L^3}\frac{|e|H}{\Gamma(-1/2+\delta)}\left\{n'\int_{0^+}^{\infty} d\eta\ \eta^{-1/2+\delta}\mathrm{e}^{-(an')^2\eta}\right.\\ &\times\left.\left(\frac{1}{|e|H\eta}+\frac{|e|H\eta}{3}+\Sigma\right)\right\}_{n'\to\infty} - \frac{2^4(2\pi)^5}{720L^7}\frac{|e|H}{\Gamma(-1/2+\delta)}\\ &\left\{n'3\int_{0^+}^{\infty} d\eta\ \eta^{3/2+\delta}e^{-(an')^2\eta}\left(\frac{1}{|e|H\eta}+\frac{|e|H\eta}{3}+\Sigma\right)\right\}_{n'\to\infty}\\ &+\frac{12(2\pi)^3}{720L^5}\frac{|e|H}{\Gamma(-1/2+\delta)}\left\{n'\int_{0^+}^{\infty} d\eta\ \eta^{1/2+\delta}e^{-(an')^2\eta}\right.\end{aligned}$$

$$\times\left(\frac{1}{|e|H\eta}+\frac{|e|H\eta}{3}+\Sigma\right)\Bigg\}_{n'\to\infty}. \tag{21}$$

where

$$\sum \equiv \sum_{k=2}^{\infty}\frac{2^{2k}B_k}{(2k)!}(|e|H\eta)^{2k-1}, \tag{22}$$

and the B_k's are the Bernoulli numbers.

We are now in position to perform, term by term in the expansion of integral (21), the limit $n'\to\infty$. After a straightforward calculation we obtain

$$\varepsilon_0' = \frac{|e|H}{8\pi^2}\int_{0^+}^{\infty} d\eta\ \eta^2 \coth(|e|H\eta) - \frac{2\pi^2}{45L^4}, \tag{23}$$

where the same kind of analytic extension made in the previous sections was performed in the manipulation of the first term in the second line of (21). Again, it gives no contribution.

Finally, the energy density of the empty space may be obtained by taking the limit of zero field and infinite volume in (23). We must subtract (23) from this quantity, obtaining

$$\Delta\varepsilon_0 = -\frac{1}{8\pi^2}\int_{0^+}^{\infty}\frac{d\eta}{\eta^3}\,\{|e|H\eta\coth(|e|H\eta)-1\}+\frac{2\pi^2}{45L^4}, \tag{24}$$

which clearly shows the influence of the external magnetic field to the fermionic Casimir effect. It must be noted that the above expression recovers (9) in the limit of zero magnetic field. In addition, the first term in (24) might be recognized as the E-K-H correction to the effective Lagrangian density, which accounts for the nonlinear effects induced by the external field in effective quantum electrodynamics (Berestetskii *et al.*, 1997; Bonneau, 1990; Courant and John, 1974; Euler, 1936; Euler and Kockel, 1935; Erdélyi *et al.*, 1953; Heisenberg and Euler, 1936; Mostepanenko and Trunov, 1988; Weisskopf, 1936; Tomazelli and Costa, 2003). It provides exactly the same contribution obtained when the limit $L\to\infty$ is considered, i.e., the contribution from the boundaries just add a field-independent amount to the E-K-H effective Lagrangian density. The independence of both effects clarify the physics governing the behavior of quantum fields under the influence of external fields and/or boundaries conditions. The generalization of the above calculation to the twisted case is immediate as well as the inclusion of the fermion mass.

4. CONCLUDING REMARKS

Using an approach based on the combination of the analytic regularization method through a representation and the Euler–Maclaurin summation formula, we have re-derived the fermionic Casimir energy densities. To this end, it was considered a $S^1 \times R^3$ space topology and the role played by the fermion mass and that of an external field on the Casimir energy density were fully investigated.

As was shown in Section 3, our approach provided a powerful way to deal with, in each step of the calculation, the divergences inherent to the theory. It was found that the fermion mass does not play any influence on the twisted and untwisted fermionic Casimir energy densities, which is in contrast with the first results obtained by Ford (1980).

We have also shown that, when an external magnetic field is considered, its effect on the Casimir energy density appears as an L-independent term (which was ultimately identified with the well-known E-K-H effective Lagrangian density) plus a term identical to that obtained when the external field is absent. This result clearly shows the independence of the external field on the boundary conditions in the weak field regime.

Finally, it must be emphasized that the present construct is a simple and easily generalizable method to reexamine other phenomena. Among these are those related to the Effective Quantum Electrodynamics in the context of the "old-fashioned" Weisskopf's method (Weisskopf, 1994), recently re-addressed (Tomazelli and Costa, 2001).

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